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When Is an Operator the Integral of a Given Spectral Measure?

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For a closed densely defined operator T on a complex Hilbert space $\mathscr{H}$ and a spectral measure E for $\mathscr{H}$ of countable multiplicity q defined on a σ-algebra $\mathscr{B}$ over an arbitrary space Λ we give three conceptually differing but equivalent answers to the question asked in the title of the paper (Theorem 1.5). We then study the simplifications which accrue when T is continuous or when $q = 1$ (Sect. 4). With the aid of these results we obtain necessary and sufficient conditions for T to be the integral of the spectral measure of a given group of unitary operators parametrized over a locally compact abelian group Γ (Sect. 5). Applying this result to the Hilbert space $\mathscr{H}$ of functions which are L_2 with respect to Haar measure for Γ, we derive a generalization of Bochner's theorem on multiplication operators (Sect. 6). Some results on the multiplicity of indicator spectral measures over Γ are also obtained. When $\Gamma = \mathbb{R}$ we easily deduce the classical theorem about the commutant of the associated self-adjoint operator (Sect. 7).

Contents.

1. INTRODUCTION

Our purpose is to prove Theorem 1.5 below, which completely answers the question asked in the title of the paper for closed, densely defined operators on Hilbert spaces and spectral measures of countable multiplicities, to deduce from it certain corollaries of interest, and to apply these results to locally compact abelian groups, thereby recovering some classical results. To state the theorem succinctly, we first introduce some notation and terminology.

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1.1 DEFINITION. Let $\mathscr{F}$ be a family of continuous linear operators on a Hilbert space $\mathscr{H}$ to itself, and $F(\cdot)$ be a function whose values are such operators. (a) The *total multiplicity of* $\mathscr{F}$ is defined to be the minimum of the cardinal numbers of subsets $G \subseteq \mathscr{H}$ such that[1]

$$\mathfrak{S}\{T(G) : T \in \mathscr{F}\} = \mathscr{H}.$$

(b) The *total multiplicity of* $F(\cdot)$ is defined to be that of Range $F(\cdot)$.

1.2 *Notation.* (i) $\mathscr{H}$ is a Hilbert space over the complex number field $\mathbb{C}$.

(ii) $E(\cdot)$ is a spectral measure for $\mathscr{H}$ on a σ-algebra $\mathscr{B}$ over a space Λ, of total multiplicity $q \leqslant \aleph_0$.

(iii) $\forall x \in \mathscr{H}$, $\mathscr{S}_x =_{\mathrm{d}} \mathfrak{S}\{E(B)(x): B \in \mathscr{B}\}$; $L_x =_{\mathrm{d}}$ the orthogonal projection on $\mathscr{H}$ onto $\mathscr{S}_x$; $\mathscr{S}_x$ and L_x are called the *E-cyclic subspace* and the *E-cyclic projection due to x.*

(iv) For any linear operator T from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$, $\mathscr{S}_T =_{\mathrm{d}} \bigcup_{x \in \mathscr{D}_T} \mathscr{S}_x$.

1.3 DEFINITION. Let $\mathscr{A}$ be any nonvoid family of subsets of Λ, and ϕ be a function of Λ to $\mathbb{C}$. We say that ϕ is *$\mathscr{A}$-measurable*, iff for all Borel subsets S of $\mathbb{C} \setminus \{0\}$, $\phi^{-1}(S) \in \mathscr{A}$.

We shall adhere to this definition of measurability throughout this paper. Of course, when $\mathscr{A}$ is an algebra the definition is equivalent to the one obtained by replacing "$\mathbb{C} \setminus \{0\}$" by "$\mathbb{C}$."

1.4 DEFINITION. Let T be any linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$. We say that (a) T is an *E-integral*, iff $T = \int_\Lambda \phi(\lambda)\, E(d\lambda)$, where ϕ is a $\mathscr{B}$-measurable function on Λ to $\mathbb{C}$;

(b) T is *E-subordinative*, iff $\forall x \in \mathscr{D}_T$, $T(x) \in \mathscr{S}_x$;

(c) T is *restrictionwise E-commuting*, iff $\forall B \in \mathscr{B}$, $E(B) \cdot T \subseteq T \cdot E(B)$;

(d) T is *E-reducing*, iff $\forall y \in \mathscr{S}_T$, $L_y \cdot T \subseteq T \cdot L_y$;

(e) T is *E-isotropic*, iff $\forall y \in \mathscr{S}_T$, $L_y(\mathscr{D}_T) \subseteq \mathscr{D}_T$.

1.5 MAIN THEOREM. *Let* (i) $\mathscr{H}$, Λ, $\mathscr{B}$, $E(\cdot)$, q *be as* in 1.2, (ii) T *be a* (*single-valued*) *closed linear operator from* $\mathscr{H}$ *to* $\mathscr{H}$ *with domain* $\mathscr{D}_T$ e.d. *in* $\mathscr{H}$. *Then the following conditions are equivalent*:

(α) *T is an E-integral,*

(β) *T is E-subordinative and E-isotropic,*

(γ) *T is E-reducing,*

(δ) *T is E-subordinative and restrictionwise E-commuting.*

[1] For $A \subseteq \mathscr{H}$, $\mathfrak{S}(A) =_d$ the (closed linear) subspace spanned by A. $=_d$ means: equal by definition.

This theorem, a condensed version of which we announced in [17] is a generalization of the results 1.1, (3.3)–(3.5) proved in [12] for $\Lambda = \mathbb{R}$, the real number field, and of their extensions to locally compact abelian groups announced in [22]. It emerged from the gradual realization that the topological and group properties of Λ involved in these earlier results were theoretically extraneous (although germane to the applications of the results to stochastic processes and filter theory; cf., e.g., [3; 12, Sect. 2; 4; 24; 27; 28]), and that it is best to let Λ be any arbitrary set. Abandonment of the hypothesis of a topological group structure necessitates only one major change in our earlier work, viz, replacement of the condition that T is stationary or "time-invariant" by the E-commuting condition in 1.5(δ). But this change results in a simpler proof than the one for $\Lambda = \mathbb{R}$ sketched in [12]. Our earlier results of course emerge on applying 1.5 to l.c.a. groups Λ, and using a few straightforward results on unitary groups parametrized on such groups (Sect. 5).

The question asked in the title of the paper was first answered in the thirties for spectral measures E on the family $\mathscr{B}$ of Borel subsets of $\Lambda = \mathbb{R}$ by von Neumann, Riesz, and Mimura (cf., e.g., [20, p. 351]), and by Sasaki, Ogasawara and Mimura (cf. [26, p. 191]). The condition obtained was that the commutant of T contain the commutant of the self-adjoint operator $H = \int_\Lambda \lambda E(d\lambda)$. This condition has of course no analog for arbitrary Λ, since the last integral becomes meaningless. On the other hand, as indicated in [12, Sect. 3] and shown below (Sect. 7), this commutant condition follows trivially from the $(\gamma) \Rightarrow (\alpha)$ part of Theorem 1.5 when $\Lambda = \mathbb{R}$. Theorem 1.5 provides three conceptually differing but equivalent answers to our question in the most general setting, reveals the full picture, and immediately yields important corollaries (Sects. 4–7). Some counterexamples (Sect. 4) also show that it is in a sense the best possible result of its kind. The natural question as to what happens when one or more conditions in the theorem are relaxed suggests further research. For instance, removal of the E-subordinative condition yields extensions of 1.5 in which ϕ is operator-valued. These extensions can be framed in terms of quasi-isometric measures (cf. [16]), or in terms of the inflated Hilbert modules considered in [21, 23].

It is convenient to break up the proof of Theorem 1.5 into several lemmas, some being interesting in themselves. These lemmas are given in Section 2. In Section 3 we prove the theorem itself. Some corollaries of interest are deduced in Section 4. The results for locally compact abelian groups, published or announced earlier, are recovered in Section 5. In Section 6 we deduce from these results a generalized version of a theorem of Bochner on multiplication operators [1]. Finally, in Section 7 we deduce the classical commutant condition for spectral measures over $\mathbb{R}$ alluded to in the last paragraph.

In the organization of the paper our results are ordered by decreasing generality. Results with multiplicity $q \geqslant 1$ precede those with $q = 1$, and are not derived from the latter by piecing. Likewise, results for arbitrary Λ precede those for more familiar spaces, and are not derived from the latter by mappings. A reverse development in which the general results are derived from the classical ones for $\mathbb{R}$ is possible. But we believe that the maintenance of an order of descending generality is more revealing and more suggestive in the present situation.

2. Ancillary Results

In this section we adopt the Notation 1.2 and also the following:

$$\forall x, y \in \mathscr{H} \ \& \ \forall B \in \mathscr{B}, \qquad \xi_x(B) \underset{\mathrm{d}}{=} E(B)x, \qquad \mu_{xy}(B) \underset{\mathrm{d}}{=} (\xi_x(B), \xi_y(B)). \tag{2.1}$$

Obviously μ_{xy} is a bounded complex-valued c.a. measure on $\mathscr{B}$. By the Schwarz inequality,

$$|\mu_{xy}(B)|^2 \leqslant \mu_{xx}(B) \cdot \mu_{yy}(B),$$

and so $\mu_{xy} \ll \mu_{xx} \ \& \ \mu_{yy}$.[2] Also, *$\xi_x$ is a bounded, $\mathscr{H}$-valued c.a.o.s. measure on $\mathscr{B}$ with nonnegative measure μ_{xx}*, in the sense of [13, Definitions 1.2, 1.4]. It follows from the integration theory developed in [13, cf. Definitions 5.4, 5.6], that

$$\forall x \in \mathscr{H}, \qquad \forall f \in L_2(\Lambda, \mathscr{B}, \mu_{xx}\,; \mathbb{C}), \qquad \int_x f(\lambda)\, \xi_x(d\lambda) \in \mathscr{H}. \tag{2.2}$$

This integration theory provides an efficient way to define and study the operator-valued E-integrals of Stone and von Neumann alluded to in 1.4(a):

2.3 Definition. Let ϕ be any (not necessarily bounded) $\mathscr{B}$-measurable function on Λ to $\mathbb{C}$ (cf. 1.3). Then $\int_\Lambda \phi(\lambda)\, E(d\lambda)$ is defined to be the operator T whose domain is given by

$$\mathscr{D}_T = \{x : x \in \mathscr{H} \ \& \ \phi \in L_2(\Lambda, \mathscr{B}, \mu_{xx}\,; \mathbb{C})\}$$

[2] $\ll$ refers to the relation of absolute continuity.

and such that

$$\forall x \in \mathscr{D}_T, \qquad T(x) = \int_\Lambda \phi(\lambda)\, \xi_x(d\lambda), \qquad \text{cf. (2.2).}$$

By a standard argument we can prove that the T in 2.3 is a normal operator from $\mathscr{H}$ to $\mathscr{H}$, i.e., a closed, linear operator with domain e.d. in $\mathscr{H}$ and such that $TT^* = T^*T$. It is easily seen that for bounded ϕ, our E-integral reduces to the one defined by Halmos [6, p. 60] and that for $\Lambda = \mathbb{R}$ and $\mathscr{B} =$ the Borel family over $\mathbb{R}$, it reduces to the well-known, standard integral $T(F)$ given in Stone [25, 6.1 *et seq.*].

It is convenient at this stage to recall the concept of the *strong limit* of any sequence of operators:

2.4 DEFINITION. Let $(T_n)_1^\infty$ be any sequence of operators T_n from $\mathscr{H}$ to $\mathscr{H}$ and let $\mathscr{D}_{T_n}$ be the domain of T_n. Then $\text{slim}_{n\to\infty} T_n$ is defined as the operator T whose domain is given by

$$\mathscr{D}_T = \{x : x \in \lim_{n\to\infty} \mathscr{D}_{T_n} \ \&\ \lim_{n\to\infty} T_n(x) \text{ exists}\}$$

and such that $\forall x \in \mathscr{D}_T$, $T(x) = \lim_{n\to\infty} T_n(x)$.

In case the T_n are linear, then obviously so is their strong limit T, and $\{0\} \subseteq \mathscr{D}_T \subseteq \mathscr{H}$. We leave to the reader the proof of the following simple result on strong limits.

2.5 LEMMA.[3] *Let S and T_n, $n \in \mathbb{N}_+$, be any* (s.v.) *linear operators from $\mathscr{H}$ to $\mathscr{H}$. Then*

(a) $$\underset{n\to\infty}{\text{slim}}(T_n \cdot S) = (\underset{n\to\infty}{\text{slim}}\, T_n) \cdot S;$$

(b) *When S is closed and* $\text{slim}_{n\to\infty} T_n$ *has domain $\mathscr{H}$,*

$$\underset{n\to\infty}{\text{slim}}(S \cdot T_n) \subseteq S \cdot \underset{n\to\infty}{\text{slim}}\, T_n;$$

(c) *When S is continuous on $\mathscr{H}$,*

$$\underset{n\to\infty}{\text{slim}}(S \cdot T_n) \supseteq S \cdot \underset{n\to\infty}{\text{slim}}\, T_n.$$

We now turn to the specific lemmas needed to prove Theorem 1.5. Our first result records some obvious properties of cyclic projections and subspaces. For the proofs see Halmos [6, p. 91, Theorems 1, 3; p. 92, Theorem 4].

[3] $\mathbb{N}$ and $\mathbb{N}_+$ refer to the sets of integers and positive integers, respectively.

2.6 LEMMA.

(a) $\forall B \in \mathscr{B}$ & $\forall x \in \mathscr{H}$, $\quad E(B) \cdot L_x = L_x \cdot E(B) = L_{E(B)x}$.

(b) $\forall x, y \in \mathscr{H}$, $\quad y \in \mathscr{S}_x \Rightarrow \mathscr{S}_y \subseteq \mathscr{S}_x$,

$$y \in \mathscr{S}_x^{\perp} \Rightarrow \mathscr{S}_y \subseteq \mathscr{S}_x^{\perp}.$$

2.7 LEMMA (on cyclic projections). *Let $y, z \in \mathscr{H}$. Then*

$$\frac{d\mu_{zy}}{d\mu_{yy}} \in L_2(\Lambda, \mathscr{B}, \mu_{yy}\,;\, \mathbb{C}) \;\&\; L_y(z) = \int_\Lambda \frac{d\mu_{zy}}{d\mu_{yy}}(\lambda)\, E(d\lambda) y.$$

Proof. This is a generalization for arbitrary σ-algebras $\mathscr{B}$ of an amended version of Kolmogorov's Theorem 8 on subordinate sequences, in which $\mathscr{B}$ is the σ-algebra of Borel subsets of $[-\pi, \pi]$, cf. [9, Sect. 4]. It may also be viewed as a special case of [13, Projection Theorem 5.10]. ∎

Next we turn to spectral integrals.

2.8 THEOREM. *Let* (i) $R = \int_\Lambda \phi(\lambda)\, E(d\lambda)$, *where ϕ is any $\mathscr{B}$-measurable function on Λ to $\mathbb{C}$,* (ii) *K be a restrictionwise E-commuting operator, continuous on $\mathscr{H}$. Then $K \cdot R \subseteq R \cdot K$.*

Proof. Case 1. Let ϕ be $\mathscr{B}$-simple, i.e., $\phi = \sum_1^r a_k \chi_{A_k}$, where $a_k \in \mathbb{C}$ and the $A_k \in \mathscr{B}$ are disjoint. Then it follows at once from Definition 2.3 and [13, 5.4] that $R = \sum_1^r a_k E(A_k)$. Hence by (ii) we obviously have $K \cdot R = R \cdot K$.

Case 2. Let ϕ be any $\mathscr{B}$-measurable function on Λ to $\mathscr{C}$. Then there exists a sequence $(\phi_n)_1^\infty$ of $\mathscr{B}$-simple functions ϕ_n such that

$$\forall \lambda \in \Lambda, \qquad \phi_n(\lambda) \to \phi(\lambda) \;\&\; |\phi_n(\lambda)| \leqslant |\phi(\lambda)|.$$

Let $R_n =_{\mathrm{d}} \int_\Lambda \phi_n(\lambda)\, E(d\lambda)$. Then it follows readily from Definition 2.3 [13, 5.6] and Definition 2.4 that $R = \operatorname{slim}_{n\to\infty} R_n$. Since by Case 1, $K \cdot R_n = R_n \cdot K$, and K is continuous on $\mathscr{H}$, we conclude from Lemma 2.5 that

$$\begin{aligned} K \cdot R &= K \cdot \operatorname*{slim}_{n\to\infty} R_n \\ &\subseteq \operatorname{slim}(K \cdot R_n) = \operatorname*{slim}_{n\to\infty}(R_n \cdot K) = (\operatorname*{slim}_{n\to\infty} R_n)K = R \cdot K. \end{aligned}$$ ∎

2.9 LEMMA (on spectral integrals). *Let*

(i) $R = \int_A \phi(\lambda)\, E(d\lambda)$, *where* ϕ *is* $\mathscr{B}$*-measurable on* A *to* $\mathbb{C}$;

(ii) $\forall n \in \mathbb{N}_+,\ A_n =_{\mathrm{d}} \{\lambda\colon |\phi(\lambda)| \leqslant n\}$.

Then

(a) *R is restrictionwise E-commuting*;

(b) $\forall x \in \mathscr{H}, L_x(\mathscr{D}_R) \subseteq \mathscr{D}_R, L_x \cdot R \subseteq R \cdot L_x$ *and so R is E-isotropic and E-reducing*;

(c) *R is E-subordinative*;

(d) $\forall n \in \mathbb{N}_+$, $R \cdot E(A_n)$ *is continuous on* $\mathscr{H}$, *and*

$$\operatorname*{slim}_{n\to\infty} E(A_n) = I \;\&\; \operatorname*{slim}_{n\to\infty} R \cdot E(A_n) = R.$$

Proof. Taking $K = E(B)$ in Theorem 2.8, we get (a). Taking $K = L_x$ in 2.8, and recalling 2.6(a), we get (b). Property (c) is obvious from the definition of a spectral integral. (d) Obviously $R \cdot E(A_n) = \int_A \phi(\lambda)\, \chi_{A_n}(\lambda)\, E(d\lambda)$. Hence $|R \cdot E(A_n)| = E\text{-ess sup}\, |\phi \cdot \chi_{A_n}| \leqslant n$, and so $R \cdot E(A_n)$ is continuous on $\mathscr{H}$. The first equality in (d) is utterly obvious; the second is a simple consequence of "Lebesgue's Dominated Convergence Theorem" for spectral integrals, since

$$|\phi(\lambda)\, \chi_{A_n}(\lambda)| \leqslant |\phi(\lambda)| \;\&\; \phi(\lambda)\, \chi_{A_n}(\lambda) \to \phi(\lambda), \qquad \text{as} \quad n \to \infty. \quad \blacksquare$$

We turn next to operators of the type considered in Theorem 1.5.

2.10 LEMMA. *Let* (i) *T be a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$*,

(ii) $\mathscr{C} = \{S\colon S$ *is an E-integral, continuous on* $\mathscr{H}\}$.

Then

(a) *T is restrictionwise E-commuting* $\Leftrightarrow \forall S \in \mathscr{C},\ S \cdot T \subseteq T \cdot S$.

(b) *T is restrictionwise E-commuting* & $S \in \mathscr{C} \Rightarrow \forall x \in \mathscr{D}_T$, $S(x) \in \mathscr{D}_T \cap \mathscr{S}_x$.

Proof. (a) Suppose that $\forall S \in \mathscr{C},\ S \cdot T \subseteq T \cdot S$, and let $B \in \mathscr{B}$. Obviously $S =_{\mathrm{d}} E(B) \in \mathscr{C}$, and therefore $E(B) \cdot T \subseteq T \cdot E(B)$. Thus T is restrictionwise E-commuting.

Conversely, suppose that

$$T \text{ is restrictionwise } E\text{-commuting}, \tag{1}$$

and let $S \in \mathscr{C}$, say $S = \int_A \psi(\lambda)\, E(d\lambda)$, where ψ is $\mathscr{B}$-measurable on A to $\mathbb{C}$. Then by the definition of spectral integral,

$$S = \operatorname*{slim}_{n\to\infty} S_n, \qquad \text{where} \quad S_n = \int_A \psi_n(\lambda)\, E(d\lambda). \tag{2}$$

and $(\psi_n)_1^\infty$ is a suitable sequence of $\mathscr{B}$-simple functions on Λ to $\mathbb{C}$.[4] Let

$$\psi_n = \sum_{k=1}^{r_n} a_{nk}\chi_{B_{nk}} \qquad \text{so that} \qquad S_n = \sum_{k=1}^{r_n} a_{nk}E(B_{nk}).$$

Then from (1) it follows easily that

$$\forall n \in \mathbb{N}_+, \qquad S_n \cdot T \subseteq T \cdot S_n. \tag{3}$$

Now by (2) and Lemma 2.5(a),

$$S \cdot T = (\operatorname*{slim}_{n\to\infty} S_n) \cdot T = \operatorname*{slim}_{n\to\infty}(S_n \cdot T). \tag{4}$$

But by (i), T is closed, and $\mathscr{D}_S = \mathscr{H}$. Hence by Lemma 2.5(b) and (2),

$$\operatorname*{slim}_{n\to\infty}(TS_n) \subseteq T \cdot \operatorname*{slim}_{n\to\infty} S_n = T \cdot S. \tag{5}$$

From (3), (4), (5), $S \cdot T \subseteq T \cdot S$. This finishes the proof of (a). Part (b) follows immediately from (a). ▮

The next lemma is crucial for our proof of the hard implication $(\delta) \Rightarrow (\alpha)$ in Theorem 1.5.

2.11 Main Lemma.[5] *Let* (i) *T be a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$* e.d. *in $\mathscr{H}$*, (ii) *T be E-subordinative and restriction-wise E-commuting. Then*

$$\forall y \in \mathscr{D}_T, \qquad L_y(\mathscr{D}_T) \subseteq \mathscr{D}_T \,\&\, L_y \cdot T \subseteq T \cdot L_y.$$

Proof. Let $y, z \in \mathscr{D}_T$. Then by Lemma 2.7,

$$L_y(z) = \int_\Lambda \phi(\lambda)\, E(d\lambda)(y), \qquad \text{where} \quad \phi = \frac{d\mu_{zy}}{d\mu_{yy}}. \tag{1}$$

Let

$$B_n \underset{\mathrm{d}}{=} \{\lambda : |\phi(\lambda)| \leqslant n\}, \qquad n \in \mathbb{N}_+.$$

Since by (ii), $\mathscr{D}_T = \mathscr{D}_{E(B_n)T} \subseteq \mathscr{D}_{TE(B_n)}$, it follows that

$$z_n \underset{\mathrm{d}}{=} E(B_n)z \in \mathscr{D}_T, \qquad n \in \mathbb{N}_+. \tag{2}$$

[4] Specifically, the ψ_n are such that $|\psi_n(\lambda)| \leqslant |\psi(\lambda)|$ and $\psi_n(\lambda) \to \psi(\lambda)$, a.e. (E) on Λ.

[5] This lemma misses being the implication $(\delta) \Rightarrow (\gamma)$ in 1.5, since the quantifier in the conclusion is "$\forall y \in \mathscr{D}_T$" and not "$\forall y \in \mathscr{S}_T$."

By (2) and 2.6(a),

$$\hat{z}_n \underset{\mathrm{d}}{=} L_y(z_n) = L_y E(B_n)(z) = E(B_n)\, L_y(z). \tag{3}$$

Next, from (3) and (1),

$$\hat{z}_n = \int_\Lambda \chi_{B_n}(\lambda)\, \phi(\lambda)\, E(d\lambda)(y) = S_n(y),$$

say, where S_n is an E-integral which (cf. 2.9(d)) is continuous on $\mathscr{H}$. Hence by Lemma 2.10(b),

$$\hat{z}_n \in \mathscr{D}_T \cap \mathscr{S}_y . \tag{4}$$

From (2) and (4) we see that

$$z_n - \hat{z}_n = z_n - L_y(z_n) \in \mathscr{D}_T \cap \mathscr{S}_y^{\perp}. \tag{5}$$

Since T is E-subordinative, we conclude from (4), (5), and 2.6(b) that

$$T(\hat{z}_n) \in \mathscr{S}_{\hat{z}_n} \subseteq \mathscr{S}_y , \qquad T(z_n - \hat{z}_n) \in \mathscr{S}_{z_n - \hat{z}_n} \subseteq \mathscr{S}_y^{\perp}.$$

Thus

$$L_y T(\hat{z}_n) = T(\hat{z}_n) \,\&\, L_y T(z_n - \hat{z}_n) = 0;$$

whence

$$L_y T(z_n) = L_y T(z_n - \hat{z}_n) + L_y T(\hat{z}_n) = T(\hat{z}_n). \tag{6}$$

Now since T is E-commuting and $z \in \mathscr{D}_T$, therefore

$$E(B_n) T(z) = TE(B_n)(z),$$

and so from (6), (2), and 2.6(a),

$$T(\hat{z}_n) = L_y T(z_n) = L_y TE(B_n) z = L_y E(B_n)\, T(z) = E(B_n)\, L_y T(z). \tag{7}$$

Finally, since $\operatorname{slim}_{n\to\infty} E(B_n) = I$, we conclude from (3) and (7) that

$$\lim_{n\to\infty} \hat{z}_n = L_y(z) \,\&\, \lim_{n\to\infty} T(\hat{z}_n) = L_y T(z).$$

Since T is closed, it follows that

$$L_y(z) \in \mathscr{D}_T \,\&\, TL_y(z) = L_y T(z).$$

Since y, $z \in \mathscr{D}_T$ are arbitrary, we are done. ▮

3. Proof of Main Theorem 1.5

We shall prove the implications in 1.5 in order of difficulty, viz. $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \Rightarrow (\alpha)$.

The first implication is just a conjunction of the well-known results stated in 2.9(c), (b).

Next assume (β), i.e.,

$$\forall x \in \mathscr{S}_T, \qquad L_x(\mathscr{D}_T) \subseteq \mathscr{D}_T, \tag{1}$$

$$\forall y \in \mathscr{D}_T, \qquad T(y) \in \mathscr{S}_y, \tag{2}$$

and let $x \in \mathscr{H}$. Then by (2) and 2.6(b),

$$\forall y \in \mathscr{D}_T \cap \mathscr{S}_x, \qquad T(y) \in \mathscr{S}_y \subseteq \mathscr{S}_x;$$

$$\forall y \in \mathscr{D}_T \cap \mathscr{S}_x^{\perp}, \qquad T(y) \in \mathscr{S}_y \subseteq \mathscr{S}_x^{\perp}.$$

Thus

$$T(\mathscr{D}_T \cap \mathscr{S}_x) \subseteq \mathscr{S}_x \;\&\; T(\mathscr{D}_T \cap \mathscr{S}_x^{\perp}) \subseteq \mathscr{S}_x^{\perp}. \tag{3}$$

By (1) and (3), $\forall x \in \mathscr{S}_T$, the subspace $\mathscr{S}_x$ reduces T, i.e., $L_x \cdot T \subseteq T \cdot L_x$. Thus (γ).

Now assume (γ), i.e.,

$$\forall y \in \mathscr{S}_T, \qquad L_y \cdot T \subseteq T \cdot L_y, \tag{4}$$

and let $x \in \mathscr{D}_T$ and $B \in \mathscr{B}$. Then $x \in \mathscr{S}_T$ and so by (4), $L_x T(x) = TL_x(x) = T(x)$, i.e., $T(x) \in \mathscr{S}_x$. Thus T is E-subordinative. From this and 2.6(a) we get

$$E(B)\ T(x) = E(B)\, L_x T(x) = L_{E(B)x} T(x).$$

But $E(B)(x) \in \mathscr{S}_T$, and so by (4), $L_{E(B)x} T \subseteq T L_{E(B)x}$. Hence, again using 2.6(a),

$$E(B) \cdot T(x) = T \cdot L_{E(B)x}(x) = T \cdot E(B)\, L_x(x) = T \cdot E(B)(x).$$

Thus $E(B) \cdot T \subseteq T \cdot E(B)$, i.e., T is restrictionwise E-commuting. This establishes (δ).

We come now to the harder implication $(\delta) \Rightarrow (\alpha)$. Assume (δ). Our first task is to identify the function ϕ occurring in (α). For this, let K be the initial segment of cardinality q of the naturally ordered set $\mathbb{N}_+$ of positive integers. Then by 1.2(ii),

$$\exists \{w_k\}_{k \in K} \subseteq \mathscr{H} \ni \mathscr{H} = \mathfrak{S}\{E(B)(w_k) : B \in \mathscr{B} \;\&\; k \in K\}.$$

But since by 1.5(ii), $\mathscr{D}_T$ is e.d. in $\mathscr{H}$, therefore for each $k \in K$, there exists a sequence of vectors in $\mathscr{D}_T$ converging to w_k. It follows that there exists an initial segment L of the ordered set $\mathbb{N}_+$ such that

$$\exists \{z_l\}_{l \in L} \subseteq \mathscr{D}_T \setminus \{0\} \ni \mathscr{H} = \mathfrak{S}\{E(B)(z_l) : B \in \mathscr{B} \,\&\, l \in L\}.$$

Now, define recursively,

$$y_1 \underset{\mathrm{d}}{=} z_1 \,\&\, \forall l \in L \setminus \{1\}, \qquad y_l \underset{\mathrm{d}}{=} z_l - \sum_{k=1}^{l-1} L_{y_k}(z_l).$$

Since y_1 and $z_2 \in \mathscr{D}_T$, therefore by Lemma 2.11, $L_{y_1}(z_2) \in \mathscr{D}_T$; whence

$$y_2 \underset{\mathrm{d}}{=} z_2 - L_{y_1}(z_2) \in \mathscr{D}_T.$$

Proceeding in this way, we see that each $y_l \in \mathscr{D}_T$. After deletion of the y_l which are zero, we get $\{y_j\}_{j \in J}$, where J is an initial segment of $\mathbb{N}_+$ and the y_j are nonzero and orthogonal. Letting $x_j = y_j / |y_j|$, we conclude that

$$\mathscr{H} = \sum_{j \in J} \mathscr{S}_{x_j}, \qquad \mathscr{S}_{x_j} \perp \mathscr{S}_{x_{j'}} \text{ for } j \neq j' \,\&\, (x_j)_{j \in J} \text{ is o.n. in } \mathscr{D}_T. \tag{5}$$

From (5) and the E-subordinative condition in (δ) the $T(x_j)$, $j \in J$, are orthogonal. Hence there are sequences $(c_j)_{j \in J}$ of positive numbers such that both $\sum_{j \in J} c_j x_j$ and $\sum_{j \in J} c_j T(x_j)$ converge in $\mathscr{H}$. We could, for instance, take

$$c_j = (1/j) \min\{1, 1/|T(x_j)|\}, \qquad j \in J \qquad \text{(cf. (5))}.$$

Now, and this is crucial, let

$$x_0 \underset{\mathrm{d}}{=} \sum_{j \in J} c_j x_j, \qquad \text{so that necessarily} \quad L_{x_j}(x_0) = c_j x_j \tag{6}$$

Since as $n \to \infty$ the partial sums $\sum_{j \in J_n} c_j x_j$ and $T(\sum_{j \in J_n} c_j x_j)$, where $J_n =_{\mathrm{d}} \{j : j \in J \,\&\, j \leqslant n\}$, converge to x_0 and to some y_0 in $\mathscr{H}$, and T is closed, it follows that $x_0 \in \mathscr{D}_T$. Hence by the E-subordinative condition in (δ), $T(x_0) \in \mathscr{S}_x$; whence by Lemma 2.7,

$$\left.\begin{aligned} &T(x_0) = \int_\Lambda \phi(\lambda)\, E(d\lambda)(x_0), \\ \text{where} \quad & \\ &\phi = d\mu_{Tx_0, x_0} / d\mu_{x_0 x_0} \in L_2(\Lambda, \mathscr{B}, \mu_{x_0 x_0}; \mathbb{C}). \end{aligned}\right\} \tag{7}$$

This identifies the function ϕ. In fact, we can restate (7) in the form

$$x_0 \in \mathscr{D}_R \,\&\, R(x_0) = T(x_0), \qquad \text{where} \quad R \underset{\mathrm{d}}{=} \int_\Lambda \phi(\lambda)\, E(d\lambda). \tag{8}$$

We now complete the proof of (α) by showing that $T = R$. We first assert that

$$\forall B \in \mathscr{B} \,\&\, \forall j \in J, \qquad R \cdot E(B)(x_j) = T \cdot E(B)(x_j). \tag{I}$$

Proof of (I). From (6), (8), and Lemma 2.9(b),

$$\forall j \in J, \qquad x_j = L_{x_j}(x_0/c_j) \in \mathscr{D}_R\,. \tag{9}$$

But since $\mathscr{D}_R = \mathscr{D}_{L_{x_j}R}$ and by 2.9(b), $L_{x_j} \cdot R \subseteq R \cdot L_{x_j}$, it also follows from (8) and (6) that

$$L_{x_j} \cdot R(x_0) = R \cdot L_{x_j}(x_0) = R(c_j x_j) = c_j R(x_j).$$

Now from (8), Lemma 2.11, and (6),

$$L_{x_j} R(x_0) = L_{x_j} T(x_0) = T L_{x_j}(x_0) = T(c_j x_j) = c_j T(x_j).$$

Since $c_j \neq 0$, we conclude that

$$\forall j \in J, \qquad R(x_j) = T(x_j). \tag{10}$$

Next by Lemma 2.9(a), $E(B)(\mathscr{D}_R) \subseteq \mathscr{D}_R$. Hence from (9), $E(B)(x_j) \in \mathscr{D}_R$. It follows from 2.9(a), (10), and the E-commuting condition in (δ) that $\forall B \in \mathscr{B}$ & $\forall j \in J$,

$$RE(B)(x_j) = E(B)\, R(x_j) = E(B)\, T(x_j) = TE(B)\, x_j\,.$$

Thus (I).

Now let

$$A_n \underset{\mathrm{d}}{=} \{\lambda : |\phi(\lambda)| \leqslant n\}, \qquad n \in \mathbb{N}_+\,. \tag{11}$$

Then we assert that

$$\forall n \in \mathbb{N}_+\,, \qquad R \cdot E(A_n) = T \cdot E(A_n) \qquad \text{on} \quad \mathscr{H}. \tag{II}$$

Proof of (II). Consider an element

$$h_r \underset{\mathrm{d}}{=} \sum_{j=1}^{r} \sum_{k=1}^{m} a_{jk} E(B_{jk})(x_j) \in \sum_{j=1}^{r} \mathscr{S}_{x_j}\,. \tag{12}$$

Since $E(A_n)\,E(B_{jk}) = E(A_n \cap B_{jk})$, it follows readily from (1) that

$$\forall n,\ r \in \mathbb{N}_+ \ \&\ \forall h_r \text{ as in (12)},\ R \cdot E(A_n)(h_r) = T \cdot E(A_n)(h_r). \tag{13}$$

Now let $x \in \mathscr{H}$. Then the decomposition (5) shows that

$$x = \lim_{r\to\infty} h_r\,, \qquad \text{where} \quad h_r \text{ is of type (12).}$$

Clearly,

$$\lim_{r\to\infty} E(A_n)(h_r) = E(A_n)(x);$$

and by (13) and the continuity of $R \cdot E(A_n)$, cf. 2.9(d),

$$\lim_{r\to\infty} TE(A_n)(h_r) = \lim_{r\to\infty} R \cdot E(A_n)(h_r) = R \cdot E(A_n)(x).$$

Since T is closed, we conclude that

$$E(A_n)(x) \in \mathscr{D}_T \ \&\ TE(A_n)(x) = RE(A_n)(x).$$

This holds $\forall x \in \mathscr{H}$; hence (II).

Finally, we let $n \to \infty$ in (II). From Lemmas 2.9(d), 2.5(a), the E-commuting condition in (δ) and (II), we get

$$\begin{aligned} T &= \{\operatorname*{slim}_{n\to\infty} E(A_n)\}T = \operatorname*{slim}_{n\to\infty}\{E(A_n)\cdot T\} \\ &\subseteq \operatorname*{slim}_{n\to\infty}\{TE(A_n)\} = \operatorname*{slim}_{n\to\infty}\{RE(A_n)\} = R. \end{aligned} \tag{14}$$

By (14), $R = \operatorname{slim}_{n\to\infty}\{TE(A_n)\}$. But since T is closed and $\operatorname{slim}_{n\to\infty} E(A_n)$ has domain $\mathscr{H}$, we can apply Lemma 2.5(b) to conclude that

$$R = \operatorname*{slim}_{n\to\infty}\{T\cdot E(A_n)\} \subseteq T \cdot \operatorname*{slim}_{n\to\infty} E(A_n) = T. \tag{15}$$

By (14) and (15), $T = R$. Thus (α). This completes the proof. ∎

4. Continuous Operators and Spectral Measures with Multiplicity One

Let the closed operator T from $\mathscr{H}$ to $\mathscr{H}$ with e.d. domain $\mathscr{D}_T$ be continuous. Then $\mathscr{D}_T = \mathscr{H}$ and obviously, *T is E-isotropic for every spectral measure E for $\mathscr{H}$* (cf. 1.4(e)). From the equivalence (β) $\Leftrightarrow$ (δ) of Theorem 1.5 we may therefore conclude that

T is E-subordinative $\Rightarrow$ T is restrictionwise E-commuting.

Consequently, the clause "T is restrictionwise E-commuting" in (δ) is redundant. Hence for continuous T on $\mathscr{H}$ our Theorem 1.5 may be rendered as follows.

4.1 COROLLARY. *Let* (i) $\mathscr{H}$, Λ, $\mathscr{B}$, $E(\cdot)$, q *be as in* 1.2, (ii) T *be a continuous linear operator on* $\mathscr{H}$ *to* $\mathscr{H}$. *Then the following conditions are equivalent*:

(α) *T is an E-integral,*

(β) *T is E-subordinative,*

(γ) *T is E-reducing.*

We consider next spectral measures E for which the multiplicity $q = 1$. For this we need the following lemma, which has an interest of its own.

4.2 LEMMA (on subordinate cyclic projections). *Under the Notation* 1.2 *and* (2.1) *we have*

$$\forall x, y \in \mathscr{H}, \qquad y \in \mathscr{S}_x \Leftrightarrow \exists C \in \mathscr{B} \ni L_y = E(C) \cdot L_x .$$

Proof. Suppose that $L_y = E(C) \cdot L_x$, where $C \in \mathscr{B}$. Then obviously $y = L_y(y) = E(C) L_x(y) \in E(C)(\mathscr{S}_x) \subseteq \mathscr{S}_x$, i.e., $y \in \mathscr{S}_x$.

Conversely, let $y \in \mathscr{S}_x$. Then by Lemma 2.7,

$$y = \int_\Lambda \phi(\lambda)\, E(d\lambda)(x), \qquad \text{where} \quad \phi \underset{\mathrm{d}}{=} d\mu_{yx}/d\mu_{xx}, \tag{1}$$

from which we easily infer the Kolmogorov conditions [9, Theorem 8],

$$\mu_{yy} \ll \mu_{xx} \;\&\; (d\mu_{yy}/d\mu_{xx})(\lambda) = |\phi(\lambda)|^2, \qquad \text{a.e. } \mu_{xx} . \tag{2}$$

Now let $z \in \mathscr{H}$. Then by Lemma 2.7 and (1),

$$L_y(z) = \int_\Lambda \frac{d\mu_{zy}}{d\mu_{yy}}(\lambda)\, E(d\lambda)(y) = \int_\Lambda \frac{d\mu_{zy}}{d\mu_{yy}}(\lambda) \cdot \phi(\lambda)\, E(d\lambda)(x). \tag{3}$$

Now grant for a moment that

$$\left.\begin{aligned} &(d\mu_{zy}/d\mu_{yy})(\lambda) \cdot \phi(\lambda) = \chi_S(\lambda)(d\mu_{zx}/d\mu_{xx})(\lambda), \qquad \text{a.e. } \mu_{xx}, \\ &\text{where} \\ &S \underset{\mathrm{d}}{=} \text{the support of } \phi. \end{aligned}\right\} \tag{I}$$

Then from (3), (I), and 2.7,

$$L_y(z) = \int_\Lambda \chi_S(\lambda) \frac{d\mu_{zx}}{d\mu_{xx}}(\lambda)\, E(d\lambda)(x)$$

$$= E(S) \int_\Lambda \frac{d\mu_{zx}}{d\mu_{xx}}(\lambda)\, E(d\lambda)(x) = E(S)\, L_x(z),$$

i.e., we get the desired equality $L_y = E(C) \cdot L_x$ with $C = S$. To complete the proof we have only to verify (I).

Proof of (I). First, let $\lambda \notin S$. Then both sides of (I) vanish and we are done. Next, let $\lambda \in S$. Then the equality in (I) reduces to

$$(d\mu_{zy}/d\mu_{yy})(\lambda) \cdot \phi(\lambda) = (d\mu_{zx}/d\mu_{yy})(\lambda), \qquad \text{a.e. } \mu_{xx} \text{ on } S. \tag{I$'$}$$

To verify this, we note by (1) and the operational calculus that $\forall B \in \mathscr{B}$,

$$\mu_{zy}(B) \underset{\mathrm{d}}{=} (E(B)(z), E(B)(y)) = \left(\int_B E(d\lambda)(z), \int_B \phi(\lambda)\, E(d\lambda)(x)\right)$$

$$= \int_B \overline{\phi(\lambda)}\, \mu_{zx}(d\lambda) = \int_B \overline{\phi(\lambda)} \frac{d\mu_{zx}}{d\mu_{xx}}(\lambda)\, \mu_{xx}(d\lambda).$$

It follows that

$$(d\mu_{zy}/d\mu_{xx})(\lambda) = \overline{\phi(\lambda)}(d\mu_{zx}/d\mu_{xx})(\lambda), \qquad \text{a.e. } \mu_{xx} \text{ on } \Lambda;$$

whence, since $\lambda \in S$ and $\overline{\phi(\lambda)} \neq 0$,

$$(d\mu_{zx}/d\mu_{xx})(\lambda) = (d\mu_{zy}/d\mu_{xx})(\lambda)/\overline{\phi(\lambda)}, \qquad \text{a.e. } \mu_{xx} \text{ on } S. \tag{4}$$

Again since $\phi(\lambda) \neq 0$, neither side of (2) vanishes, and (2) can be restated as

$$1/\overline{\phi(\lambda)} = (d\mu_{xx}/d\mu_{yy})(\lambda) \cdot \phi(\lambda), \qquad \text{a.e. } \mu_{xx} \text{ on } S. \tag{5}$$

From (4) and (5),

$$\frac{d\mu_{zx}}{d\mu_{xx}}(\lambda) = \frac{d\mu_{zy}}{d\mu_{xx}}(\lambda) \cdot \frac{d\mu_{xx}}{d\mu_{yy}}(\lambda) \cdot \phi(\lambda) = \frac{d\mu_{zy}}{d\mu_{yy}}(\lambda) \cdot \phi(\lambda),$$

a.e. μ_{xx} on S, i.e., we have (I$'$). This proves (I) and therefore the lemma. ∎

Now let $q = 1$, and α be a "cyclic vector," i.e., one for which $\mathscr{S}_\alpha = \mathscr{H}$ and so $L_\alpha = I$. Taking $x = \alpha$ in 4.2 we get

$$\forall y \in \mathscr{H}, \exists C \in \mathscr{B} \ni L_y = E(C).$$

Thus every cyclic projection is a spectral one. On the other hand, given a $B \in \mathscr{B}$, we see from 2.6(a) that

$$L_{E(B)\alpha} = E(B)\, L_\alpha = E(B).$$

Thus every spectral projection is a cyclic one. Briefly,

$$\{L_x : x \in \mathscr{H}\} = \{E(B) : B \in \mathscr{B}\}, \qquad \text{for} \quad q = 1. \tag{4.3}$$

When $q = 1$, it follows from (4.3) that

T is restrictionwise E-commuting $\Rightarrow$ T is E-reducing.

From the equivalence $(\gamma) \Leftrightarrow (\delta)$ of Theorem 1.5 we may therefore conclude that *T is restrictionwise E-commuting $\Rightarrow$ T is E-subordinative.* It is the clause "T is E-subordinative" in (δ) which is now redundant. Accordingly, for $q = 1$, our Theorem 1.5 may be rendered as follows.

4.4 Corollary. *Let* (i) *and* (ii) *be as in Theorem* 1.5 *but with* $q = 1$. *Then the following conditions are equivalent.*

(α) *T is an E-integral,*

(β) *T is restrictionwise E-commuting.*

We shall now give two examples to show that the hypotheses of our corollaries cannot be weakened.

4.5 Example. Corollary 4.1 fails for closed densely defined but discontinuous operators T from $\mathscr{H}$ to $\mathscr{H}$ even when $q = 1$.

Solution. Let

$$\left.\begin{aligned} &\mathscr{H} = L_2(C), \textit{ where } C \textit{ is the additive group of real numbers, } \bmod 2\pi \textit{ with Haar measure;}\\ &\mathscr{D} = \{f : f \in \mathscr{H},\ f \textit{ is absolutely continuous on } C,\ f' \in C, \textit{ and } f(0) = 0\};\\ &T = iD, \textit{ where } D \textit{ is the restriction to } \mathscr{D} \textit{ of the differentiation operator.}\end{aligned}\right\} \tag{1}$$

Now, since 0 and 2π are equal in C, we have $f(0) = f(2\pi)$. We can therefore apply Stone's Theorem 10.7 [25, p. 428], and assert that

T is a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}$ e.d. *in $\mathscr{H}$*, (2)

and that T is symmetric but not self-adjoint and therefore nonnormal. Hence

$$T \textit{ is not an } E\textit{-integral for any spectral measure } E. \tag{3}$$

Now let τ_h be group-translation though $h \in C$, so that

$$\{(\tau_h(f)\}(t) = f(t + h), \qquad f \in \mathscr{H}, \qquad h, t \in C.$$

Then it is well known that

$$T = iD \subseteq i \cdot \operatorname*{slim}_{h \to 0} \frac{1}{h} \{\tau_h - I\};$$

and so

$$\forall f \in \mathscr{D}, \qquad T(f) = \lim_{h \to 0} i \cdot \frac{1}{h} \{\tau_h(f) - f\} \in \mathfrak{S}\{\tau_h(f) : h \in C\}. \tag{4}$$

To complete the solution we need some simple results from Section 6, to which this example belongs, logically speaking. As noted in (6.4), $(\tau_h : h \in C)$ is a strongly continuous group of unitary operators on $\mathscr{H}$ onto $\mathscr{H}$. Hence by the Generalized Stone Theorem, cf. 5.3(a),

$$\tau_h = \int_{\mathbb{N}} e^{inh} F(dn), \qquad h \in C,$$

where $\mathbb{N} =_{\mathrm{d}}$ the set of integers, i.e., the character group of C, and where $F(\cdot)$ is a spectral measure for $\mathscr{H}$ on the σ-algebra $2^{\mathbb{N}}$. Also by Corollary 5.5,

$$\mathfrak{S}\{\tau_h(f) : h \in C\} = \mathfrak{S}\{F(B)(f) : B \subseteq \mathbb{N}\} = \mathscr{S}_f\,. \tag{5}$$

By (4) and (5),

$$T \textit{ is } F\textit{-subordinative}. \tag{6}$$

Note that by 6.8(a) this spectral measure $F(\cdot)$ has total multiplicity $q = 1$. Despite this, Corollary 4.1 fails for T as shown by (3) and (6). ▮

4.6 Example. Corollary 4.4 fails for $q > 1$, even when the operator T is continuous on $\mathscr{H}$.

Solution. Let $E(\cdot)$ be any spectral measure for $\mathscr{H}$ for which $q > 1$. Then obviously,

$$\exists \alpha, \beta \in \mathscr{H} \ni L_\alpha L_\beta \neq L_\beta L_\alpha\,. \tag{1}$$

Let $T =_{\mathrm{d}} L_\alpha$. Then by Lemma 2.6(a), T is E-commuting. But by (1) and Lemma 2.9(b), T is not an E-integral. Thus Corollary 4.4 fails for T, even though T is continuous on $\mathscr{H}$. ▮

Finally we give an example to show that the crucial part $(\delta) \Rightarrow (\alpha)$ Theorem 1.5 fails without the requirement $q \leqslant \aleph_0$ on the multiplicity of $E(\cdot)$. In the classical literature, such failure of E-integrability is usually ascribed to the nonseparability of $\mathscr{H}$ (cf. e.g., Nakano [19]).

4.7 Example. Let $\mathscr{H} = L_2(\mathbb{R}, \mathscr{B}, \mu; \mathbb{C})$, where

$$\begin{aligned} \mathscr{B} &\underset{\mathrm{d}}{=} \{B : B \text{ or } \mathbb{R} \setminus B \text{ is a countable subset of } \mathbb{R}\}; \\ \forall B \in \mathscr{B}, \qquad \mu(B) &\underset{\mathrm{d}}{=} \text{the cardinal no. of } B. \end{aligned} \tag{1}$$

Also let

$$\begin{aligned} \forall B \in \mathscr{B} \mathbin{\&} \forall f \in \mathscr{H}, \qquad E(B)(f) &\underset{\mathrm{d}}{=} \chi_B \cdot f; \\ \forall f \in \mathscr{H}, \qquad T(f) &\underset{\mathrm{d}}{=} \chi_A \cdot f, \qquad \text{where} \quad A \in 2^{\mathbb{R}} \setminus \mathscr{B}. \end{aligned} \tag{2}$$

Then obviously, $E(\cdot)$ *is a spectral measure for* $\mathscr{H}$ *on the* σ*-algebra* $\mathscr{B}$.

Now let $f \in \mathscr{D}_T$. By (1), $f \in L_2$ with atomic measure μ. Hence $S =_{\mathrm{d}} \operatorname{supp} S$ is countable and therefore so is $A \cap S$. Thus $A \cap S \in \mathscr{B}$ and $Tf = \chi_{A \cap S} f = E(A \cap S) f \in \mathscr{S}_f$. Thus T *is* E*-subordinative.* Moreover, since $\forall B \in \mathscr{B}$ and $\forall f \in \mathscr{H}$,

$$E(B)\, T(f) = \chi_{A \cap B} \cdot f = T \cdot E(B)(f);$$

therefore T *is restrictionwise* E*-commuting*. But T is not an E-integral. For were $T = \int_{\mathbb{R}} \phi(\lambda)\, E(d\lambda)$, where ϕ is $\mathscr{B}$-measurable, it would follow (cf. Theorem 6.5(b)) that

$$\forall f \in \mathscr{H}, \qquad \chi_A \cdot f = T(f) = \phi \cdot f \qquad \text{on } \mathbb{R}. \tag{3}$$

Now by (1), for any $\lambda \in \mathbb{R}$, $f =_{\mathrm{d}} \chi_{\{\lambda\}} \in \mathscr{H}$. Hence by (3), $\chi_A = \phi$. This entails $A \in \mathscr{B}$, in contradiction to (2).

Thus T is a continuous, E-subordinative, and restrictionwise E-commuting operator which is not an E-integral. From Theorem 1.5 we can conclude that $E(\cdot)$ has multiplicity $q > \aleph_0$. This fact can of course be proved directly. Alternatively we may prove it by regarding μ as a Haar measure for the additive group $\mathbb{R}$ endowed with the discrete topology. Since this l.c.a. group is not σ-compact, Theorem 6.8(b) tells us that $q > \aleph_0$, in fact that $q =$ the cardinality of $\mathbb{R}$. ∎

Examples 4.5–4.7 show that our Corollaries 4.1, and 4.4 and our Theorem 1.5 are in a sense the best possible.

5. Stationary Operators

We shall now suppose that instead of an arbitrary set Λ and a spectral measure $E(\cdot)$ over it, we are given a locally compact abelian group Γ along with a unitary group $(U_t: t \in \Gamma)$. We shall adopt the following notation.

5.1 *Notation.* (i) Γ is an (additive, Hausdorff) locally compact abelian (l.c.a.) group.

(ii) $\hat{\Gamma}$ is the (multiplicative) character group of Γ.

(iii) $\forall t \in \Gamma$ and $\forall \lambda \in \hat{\Gamma}$,

$$[t, \lambda] \underset{\mathrm{d}}{=} \lambda(t) \in C \underset{\mathrm{d}}{=} \{z : z \in \mathbb{C}, |z| = 1\}.$$

(iv) $\mathrm{Bl}(\Gamma)$, $\mathrm{Bl}(\hat{\Gamma})$ are the Borel algebras over Γ, $\hat{\Gamma}$, i.e., the σ-rings generated by the open subsets of Γ, $\hat{\Gamma}$.

(v) $\mathscr{H}$ is a Hilbert space over $\mathbb{C}$.

(vi) $(U_t: t \in \Gamma)$ is a strongly continuous group of unitary operators on $\mathscr{H}$ onto $\mathscr{H}$, which has total multiplicity $q \leqslant \aleph_0$ (cf. 1.1).

(vii) $\forall x \in \mathscr{H}$, $\mathscr{M}_x =_{\mathrm{d}} \mathfrak{S}\{U_t(x): t \in \Gamma\}$.

We have denoted the elements of Γ by t and those of $\hat{\Gamma}$ by λ, as it is often suggestive to regard Γ as a (multidimensional) *time domain* and $\hat{\Gamma}$ as the corresponding *frequency domain.*

5.2 Definition. Let T be a linear operator from $\mathscr{H}$ to $\mathscr{H}$. We say that

(a) T is *U-subordinative*, iff $\forall x \in \mathscr{D}_T$, $T(x) \in \mathscr{M}_x$

(b) T is *U-stationary*, iff $\forall t \in \Gamma$, $U_t \cdot T = T \cdot U_t$.

The Generalized Stone Theorem can be restated in our terminology in the following augmented and convenient form.

5.3 Theorem. (a) $\exists$ *a unique spectral measure* $E(\cdot)$ *for* $\mathscr{H}$ *on* $\mathrm{Bl}(\hat{\Gamma})$ *such that*

$$\forall t \in \Gamma, \qquad U_t = \int_{\hat{\Gamma}} \lambda(t)\, E(d\lambda).$$

(b) *If K is a continuous linear operator on $\mathscr{H}$ to $\mathscr{H}$, then*

K is U-stationary $\Leftrightarrow$ K is E-commuting.

(c) *If $\mathscr{M}$ is a (closed) subspace of $\mathscr{H}$, then*

$\mathscr{M}$ reduces U_t, $\forall t \in \Gamma$ $\Leftrightarrow$ $\mathscr{M}$ reduces $E(B)$, $\forall B \in \mathrm{Bl}(\hat{\Gamma})$.

5.4 Definition. The measure $E(\cdot)$ given by 5.3(a) is called the *spectral measure* of the group $(U_t\colon t \in \Gamma)$.

5.5 Corollary. *With* $E(\cdot)$ *as in* 5.4, *we have* $\forall G \subseteq \mathscr{H}$,

$$\mathfrak{S}\{U_t(G) : t \in \Gamma\} = \mathfrak{S}\{E(B)(G) : B \in \mathrm{Bl}(\hat{\Gamma})\}.$$

Hence $(U_t\colon t \in \Gamma)$ *and* $E(\cdot)$ *have the same multiplicity* q (cf. 1.1).

Proof. We have to show (cf. 5.1(vii) and 1.2(iii)) that

$$\forall G \subseteq \mathscr{H}, \qquad \mathfrak{S}\{\mathscr{M}_x : x \in G\} = \mathfrak{S}\{\mathscr{S}_x : x \in G\}.$$

So we need only prove that $\forall x \in \mathscr{H}$, $\mathscr{M}_x = \mathscr{S}_x$.

Let $x \in \mathscr{H}$. Then by 5.3(a),

$$\forall t \in \Gamma, \qquad U_t(x) = \int_{\hat{\Gamma}} \lambda(t)\, E(d\lambda)(x) \in \mathscr{S}_x .$$

Hence

$$\mathscr{M}_x \underset{\mathrm{d}}{=} \mathfrak{S}\{U_t(x) : t \in \Gamma\} \subseteq \mathscr{S}_x . \tag{1}$$

To prove the reverse inclusion we note that $\forall t \in \Gamma$, $U_t(\mathscr{M}_x) = \mathscr{M}_x$, and therefore $\mathscr{M}_x$ reduces each U_t, $t \in \Gamma$. Hence by 5.3(c), $\mathscr{M}_x$ reduces each $E(B)$, $B \in \mathrm{Bl}(\hat{\Gamma})$. Hence

$$\forall B \in \mathrm{Bl}(\hat{\Gamma}), \qquad E(B)(x) \in E(B)(\mathscr{M}_x) \subseteq \mathscr{M}_x .$$

Hence

$$\mathscr{S}_x \underset{\mathrm{d}}{=} \mathfrak{S}\{E(B)(x) : B \in \mathrm{Bl}(\hat{\Gamma})\} \subseteq \mathscr{M}_x . \tag{2}$$

By (1) and (2), $\mathscr{M}_x = \mathscr{S}_x$. ▮

Our purpose is to find conditions under which a closed, densely defined, U-stationary operator T from $\mathscr{H}$ to $\mathscr{H}$ is an E-integral. This question is of considerable interest in the theory of linear filters, cf. [12, Sect. 2]. Our Theorem 1.5 would provide a complete answer, if we could supplant the U-stationarity requirement on T by the E-commuting condition. This replacement would be easy to justify had we a "P. Levy inversion formula" for the spectral measure $E(\cdot)$ of the group $(U_t\colon t \in \Gamma)$, of the kind announced in [15] for $\Gamma = \mathbb{R}$. Unfortunately, no one such formula valid for all l.c.a. groups is known,* and we have to adopt a less direct approach. According to Theorem 5.3(b), the replacement is valid for continuous T on $\mathscr{H}$. We can deduce from this the corresponding fact for any closed, densely defined T by utilizing its polar factorization, cf. 5.7. We need, however, the following triviality in order to handle the polar factors.

* *Note added in proof.* However, see Abstract in *Notices Amer. Math. Soc.* 22 (1975); paper to appear.

5.6 TRIVIALITY. Let (i) $H = \int_{\mathbb{R}} uF(du)$ be an s.a. operator from $\mathscr{H}$ to $\mathscr{H}$; (ii) K be a continuous linear operator on $\mathscr{H}$ to $\mathscr{H}$ such that $K \cdot H \subseteq H \cdot K$; (iii) $R = \int_{\mathbb{R}} \phi(u) F(du)$, where ϕ is a Borel measurable function on $\mathbb{R}$ to $\mathbb{C}$. Then $K \cdot R \subseteq R \cdot K$.

Proof. By the Spectral Theorem for self-adjoint operators, the conditions (i) and (ii) entail that

$$\forall A \in \mathrm{Bl}(\mathbb{R}), \qquad F(A) \cdot K \subseteq K \cdot F(A). \tag{1}$$

From (1) and Theorem 2.8 with E replaced by F, we get $K \cdot R \subseteq R \cdot K$. ∎

5.7 LEMMA. *Let T be a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$* c.d. *in $\mathscr{H}$. Then T is U-stationary $\Leftrightarrow$ T is restrictionwise E-commuting.*

Proof. Let T be restrictionwise E-commuting. Then by Theorem 2.10(a), $S \cdot T \subseteq T \cdot S$, where S is any E-integral which is continuous on $\mathscr{H}$. But by Theorem 5.3(a), $S =_{\mathrm{d}} U_t$, $t \in \Gamma$, is an E-integral which is continuous on $\mathscr{H}$. Hence $U_t \cdot T \subseteq T \cdot U_t$. Since U_t is unitary, we have actually $U_t \cdot T = T \cdot U_t$, i.e., T is U-stationary.

Conversely, let T be U-stationary. To show that T is restrictionwise E-commuting, we consider it polar factorization

$$T = S \cdot W, \qquad S \underset{\mathrm{d}}{=} (TT^*)^{1/2}, \qquad W \underset{\mathrm{d}}{=} \mathrm{Cl}(S^{\sim 1} \cdot T), \tag{1}$$

where $S^{\sim 1}$ is the generalized inverse of S (cf. [7]) and "Cl" stands for the closure. We first assert that

$$S \text{ and } W \text{ are } U\text{-stationary}. \tag{I}$$

Proof of (I). Let $t \in \Gamma$. Since $U_t^{-1} = U_{-t}$ is continuous on $\mathscr{H}$, therefore $T^* U_t = (U_{-t} \cdot T)^*$ and $U_t T^* = (T \cdot U_{-t})^*$. From this and the U-stationarity of T we infer the U-stationarity of T^*, and thence of the self-adjoint operator TT^*; thus, cf. (1),

$$S^2 = TT^* = \int_0^\infty uF(du) \quad \text{is } U\text{-stationary}, \tag{2}$$

where $F(\cdot)$ is a spectral measure on Bl $[0, \infty)$. Now

$$S \underset{\mathrm{d}}{=} (TT^*)^{1/2} = \int_0^\infty u^{1/2} F(du) \;\&\; S^{\sim 1} = \int_{0+}^\infty \frac{1}{u^{1/2}} F(du). \tag{3}$$

By (2) we can take $H = S^2$ and $K = U_t$ in Triviality 5.6 and conclude from it and (3) that *S and S^{-1} are U-stationary*. It follows of course that $S^{-1}T$, a partial isometry, and its closure W are also U-stationary. Thus (I).

Now let $J_n = [-n^{1/2}, n^{1/2}]$, $n \in \mathbb{N}_+$. Then by (2) and Lemma 2.9(d),

$$S_n \underset{\mathrm{d}}{=} S \cdot F(J_n) \quad \textit{is continuous on } \mathscr{H}, \textit{ and} \quad \operatorname*{slim}_{n\to\infty} S_n = S. \tag{4}$$

W, being a partial isometry on $\mathscr{H}$, is continuous on $\mathscr{H}$; so

$$T_n \underset{\mathrm{d}}{=} S_n \cdot W \quad \textit{is continuous on } \mathscr{H}. \tag{5}$$

Also, from Lemma 2.5(a), (4), and (1),

$$\operatorname*{slim}_{n\to\infty} T_n \underset{\mathrm{d}}{=} \operatorname*{slim}_{n\to\infty}(S_n \cdot W) = (\operatorname*{slim}_{n\to\infty} S_n)W = S \cdot W = T. \tag{6}$$

We now claim that

$$\forall n \in \mathbb{N}_+, \qquad T_n \textit{ is } U\textit{-stationary and } E\textit{-commuting}. \tag{II}$$

Proof of (II). It follows from (2) and the Spectral Theorem for the self-adjoint operator S^2 that $\forall n \in \mathbb{N}_+$, *$F(J_n)$ is U-stationary*. Hence (cf. (I)),

$$S_n = S \cdot F(J_n) \qquad \text{and} \qquad T_n \underset{\mathrm{d}}{=} S_n \cdot W \quad \textit{are } U\textit{-stationary}. \tag{7}$$

By (5), (7), and Theorem 5.3(b) with $K = T_n$, T_n is E-commuting. Thus (II).

Finally from (6) and Lemma 2.5(c) and (a) we get $\forall B \in \mathrm{Bl}(\hat{\Gamma})$,

$$\begin{aligned} E(B) \cdot T &= E(B)(\operatorname*{slim}_{n\to\infty} T_n) \subseteq \operatorname*{slim}_{n\to\infty}\{E(B) \cdot T_n\}, \\ \operatorname*{slim}_{n\to\infty}\{T_n \cdot E(B)\} &= \operatorname*{slim}_{n\to\infty} T_n \cdot E(B) = T \cdot E(B). \end{aligned} \tag{8}$$

But by (II), $E(B) \cdot T_n = T_n \cdot E(B)$. Hence from (8), $E(B) \cdot T \subseteq T \cdot E(B)$. Thus T is E-stationary. ∎

Corollary 5.5 and Lemma 5.7 enable us to derive at once from our Theorem 1.5 the following corresponding result for U-stationary and U-subordinative operators.

5.8 Theorem. *The notation being as in* 5.1, 5.4, *let T be a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$* e.d. *in $\mathscr{H}$. Then the following conditions are equivalent*:

(α) *T is an E-integral,*

(β) *T is U-subordinative and E-isotropic,*

(γ) *T is E-reducing*

(δ) *T is U-subordinative and U-stationary.*

Proof. From Corollary 5.5 and Lemma 5.7 it clearly follows that the conditions 5.8(α)–(δ) reduce to precisely the conditions 1.5(α)–(δ) for the spectral measure $E(\cdot)$ on $\mathrm{Bl}(\hat{\Gamma})$. Hence the theorem follows on applying Theorem 1.5, taking $\Lambda = \hat{\Gamma}, \mathscr{B} = \mathrm{Bl}(\hat{\Gamma})$ and $E(\cdot)$ as above. ∎

In exactly the same way we infer from our Corollaries 4.1 and 4.4 the following corollaries.

5.9 COROLLARY. *With the notation in* 5.1 *and* 5.4, *let T be a continuous linear operator on $\mathscr{H}$ to $\mathscr{H}$. Then the following conditions are equivalent*:

(α) *T is an E-integral*

(β) *T is U-subordinative*

(γ) *T is E-reducing.*

5.10 COROLLARY. *With the notation in* 5.1 *and* 5.4, *let*

(i) *$(U_t\colon t \in \Gamma)$ have multiplicity* 1, i.e., $\exists \alpha \in \mathscr{H} \ni \mathscr{M}_\alpha = \mathscr{H}$;

(ii) *T be a closed linear operator from $\mathscr{H}$ to $\mathscr{H}$ with domain $\mathscr{D}_T$* e.d. *in $\mathscr{H}$.*

Then the following conditions are equivalent:

(α) *T is an E-integral*

(β) *T is U-stationary.*

For $\Gamma = \mathbb{R}$, the results 5.8–5.10 reduce to those first established in [12]. Corollary 5.10 has been extended by Hannan [4, Theorem 3] to nonabelian locally compact groups Γ of "type I," $\hat{\Gamma}$ now being the set of all irreducible representations of Γ.

6. BOCHNER'S THEOREM ON MULTIPLICATION OPERATORS

Adhering to the notation in (5.1)(i) and (iv), let

$$\mathscr{L}_2 \underset{\mathrm{d}}{=} L_2(\Gamma, \mathrm{Bl}(\Gamma), m; \mathbb{C}), \tag{6.1}$$

where m is a Haar measure for the l.c.a. group Γ. We follow Hewitt and Ross [8, pp. 193–194, (i)–(viii)] is regard to "Haar measure"; thus the domain of m is $\mathrm{Bl}(\Gamma)$ (cf. (5.1)(iv)), and m is outer regular on $\mathrm{Bl}(\Gamma)$ but inner regular only on the topology of Γ and on the δ-ring of sets of finite m measure. It is necessary to consider this δ-ring and the σ-ring and σ-algebra affiliated with it:

$$\begin{aligned} \mathscr{B}_m(\Gamma) &\underset{\mathrm{d}}{=} \{B : B \in \mathrm{Bl}(\Gamma) \,\&\, m(B) < \infty\}, \\ \mathscr{B}(\Gamma) &\underset{\mathrm{d}}{=} \sigma\text{-ring}\{\mathscr{B}_m(\Gamma)\}, \\ \mathscr{A}(\Gamma) &\underset{\mathrm{d}}{=} \{A : A \subseteq \Gamma, \forall B \in \mathscr{B}(\Gamma) \,\&\, A \cap B \in \mathscr{B}(\Gamma)\}. \end{aligned} \tag{6.2}$$

Obviously $\mathscr{B}_m(\Gamma)$ is a δ-ring, $\mathscr{A}(\Gamma)$ is a σ-algebra, and

$$B_m(\Gamma) \subseteq \mathscr{B}(\Gamma) \subseteq \mathrm{Bl}(\Gamma) \subseteq \mathscr{A}(\Gamma). \tag{6.3}$$

$\mathscr{A}(\Gamma)$, often written $\mathscr{B}(\Gamma)^{\mathrm{loc}}$, is the family of sets which are "locally" in $\mathscr{B}(\Gamma)$. For non-σ-compact Γ, we have $\mathrm{Bl}(\Gamma) \subset \mathscr{A}(\Gamma)$ (cf. Lutzer [11]).

Our aim is to study the Hilbert space $\mathscr{L}_2$ in conjunction with the (unitary) translation group

$$(\tau_t : t \in \Gamma), \text{ where } \quad (\tau_t f)(s) \underset{\mathrm{d}}{=} f(s+t), \qquad f \in L_2\,, \quad s \in \Gamma, \tag{6.4}$$

and to deduce from Corollary 5.10 a generalized form of Bochner's Theorem [1] on the Fourier–Plancherel transform of an operator on $L_2(\mathbb{R})$ commuting with translations. For this we must first deal with the related group of multiplication by the characters of Γ. Fundamental to the study of this group and, more generally, to that of all multiplication operators is the following theorem.

6.5 Theorem (Indicator spectral measure). *Let* (i) $\bar{\mathscr{B}}$ *be a σ-ring over a set Λ and μ be a* c.a. *measure on* $\bar{\mathscr{B}}$ *to* $[0, \infty]$;

(ii) $\bar{\mathscr{B}}_\mu =_{\mathrm{d}} \{B\colon B \in \bar{\mathscr{B}} \,\&\, \mu(B) < \infty\}$, $\quad \mathscr{B} =_{\mathrm{d}}$ *σ-ring* $(\bar{\mathscr{B}}_\mu)$,

$$\mathscr{A} \underset{\mathrm{d}}{=} \mathscr{B}^{\mathrm{loc}} \underset{\mathrm{d}}{=} \{A : A \subseteq \Lambda \,\&\, \forall B \in \mathscr{B},\, A \cap B \in \mathscr{B}\};$$

(iii) $L_2 =_{\mathrm{d}} L_2(\Lambda, \bar{\mathscr{B}}, \mu; \mathbb{C})$;

(iv)[6] $\forall A \in \mathscr{A}, \quad E(A) = M_{\chi_A}$.

Then

(a) $E(\cdot)$ *is a spectral measure for* L_2 *on the σ-algebra* $\mathscr{A}$;

(b) $\forall \mathscr{A}$*-measurable* ϕ *on* Λ *to* $\mathbb{C}$, $\int_\Lambda \phi(\lambda)\, E(d\lambda) = M_\phi$;

[6] For all $\mathscr{A}$-measurable functions ϕ on Λ to $\mathbb{C}$, M_ϕ denotes the operation of multiplication by ϕ for L_2 defined by Domain $M_\phi = \{f : f \,\&\, \phi \cdot f \in L_2\}$, and $M_\phi(f) = \phi \cdot f$.

(c) $\forall f \in L_2$, $\mathscr{S}_f = \{\phi \cdot f\colon \phi$ *is* $\mathscr{A}$*-measurable on* Λ *to* $\mathbb{C}$ & $\phi \cdot f \in L_2\}$;

(d) $\forall f \in L_2$, f *is E-cyclic,* i.e., $\mathscr{S}_f = L_2$, $\Leftrightarrow \forall B \in \mathscr{B}$, $B \setminus S_f$ *is* μ*-negligible, where* S_f *is the support of* f;

(e) $E(\cdot)$ *has multiplicity* $1 \Leftrightarrow \exists \Lambda_0 \in \mathscr{B} \ni \forall B \in \mathscr{B}$, $B \setminus \Lambda_0$ *is* μ*-negligible*;

(f) $\Lambda \in \mathscr{B} \Rightarrow \mathscr{B} = \overline{\mathscr{B}} = \mathscr{A}$ *and* $E(\cdot)$ *has multiplicity* 1.

Proof. (a) Note that if f is $\mathscr{B}$-measurable and ϕ is $\mathscr{A}$-measurable, then $\phi \cdot f$ is $\mathscr{B}$-measurable. Consequently, for all $A \in \mathscr{A}$, M_{χ_A} carries L_2 into L_2. The remaining properties attributed to M_{χ_A} in (a) are easily established.

(b) follows readily from the definition of $E(\cdot)$ and standard results on spectral integration.

(c) Let $f, g \in L_2$. Then by Lemma 2.7 and (b), $g \in \mathscr{S}_f \Leftrightarrow g = \int_\Lambda \phi(\lambda)\, E(d\lambda)(f) = M_\phi(f) = \phi \cdot f$, where $\phi = d\mu_{gf}/d\mu_{ff}$ is $\mathscr{A}$-measurable on Λ to $\mathbb{C}$. Hence (c).

(d) Let f in L_2 be such that

$$\exists B_f \in \mathscr{B} \ni \mu(B_f \setminus S_f) > 0. \tag{1}$$

We shall show that $\exists g \in L_2 \setminus \mathscr{S}_f$. Since $B_f \setminus S_f \in \mathscr{B}$, we have [2, p. 5, Proposition 9]

$$B_f \setminus S_f = \bigcup_{k=1}^{\infty} B_k, \qquad \text{where} \quad B_k \in \overline{\mathscr{B}}_\mu, \tag{2}$$

and so $0 < \mu(B_f \setminus S_f) \leqslant \sum_1^\infty \mu(B_k)$. Clearly $\exists n \geqslant 1$ such that $0 < \mu(B_n) < \infty$. Obviously $g =_{\mathrm{d}} \chi_{B_n} \in L_2$. But $g = 1$ on B_n; whereas by (2), $f = 0$ on B_n. Hence we cannot have $g = \phi \cdot f$, a.e. μ, for any $\mathscr{A}$-measurable ϕ. Hence by (c), $g \notin \mathscr{S}_f$. This establishes the "$\Rightarrow$" part of (d).

Next let f in L_2 be such that $\forall B \in \mathscr{B}$, $\mu(B \setminus S_f) = 0$. Since for any g in L_2, $S_g \in \mathscr{B}$, it follows that

$$\forall g \in L_2, \qquad \mu(S_g \setminus S_f) = 0. \tag{3}$$

Now given a g in L_2, let

$$\phi =_{\mathrm{d}} g \cdot f^{\sim 1}, \qquad \text{where} \quad f^{\sim 1} =_{\mathrm{d}} \begin{cases} 1/f, & \text{on } S_f, \\ 0, & \text{on } \Lambda \setminus S_f. \end{cases}$$

Then f^{-1} is $\bar{\mathscr{B}}$-measurable, and since g is $\bar{\mathscr{B}}$-measurable so is ϕ. Moreover,

$$\phi \cdot f = \begin{cases} g, & \text{on } S_f, \\ 0, & \text{on } \Lambda \setminus S_f. \end{cases}$$

g and $\phi \cdot f$ differ on precisely the set $S_g \setminus S_f$. Hence by (3), $g = \phi \cdot f$, a.e. μ. Thus by (c), $g \in \mathscr{S}_f$. This proves the "$\Leftarrow$" part of (d).

(e) Let q be the multiplicity of $E(\cdot)$. If $q = 1$, take any E-cyclic vector f and let $\Lambda_0 = S_f$. Then by (d),

$$\Lambda_0 \in \mathscr{B} \;\&\; \forall B \in \mathscr{B}, \qquad B \setminus \Lambda_0 \text{ is } \mu\text{-negligible}, \tag{4}$$

as desired.

Next, suppose that (4) holds. Then, as in (2),

$$\Lambda_0 = \bigcup_{n=1}^{\infty} B_n, \qquad \text{where} \quad B_n \in \bar{\mathscr{B}}_\mu .$$

Let

$$b_k \underset{\mathrm{d}}{=} 1/\sqrt{\{2^k\, u(B_k)\}}, \qquad k \geqslant 1 \;\&\; f \underset{\mathrm{d}}{=} \sum_{k=1}^{\infty} b_k \chi_{B_k} .$$

Then obviously $f \in L_2$ and $S_f = \Lambda_0$. Hence (4) yields $\forall B \in \mathscr{B}$, $B \setminus S_f$ is μ-negligible, whence by (d), f is E-cyclic, and so $q = 1$. This completes the proof of (e).

(f) We first note that in general,

$$\mathscr{B} \subseteq \bar{\mathscr{B}} \subseteq \bar{\mathscr{B}}_\mu^{\mathrm{loc}} = \mathscr{A}. \tag{5}$$

The second inclusion in (5) is clear from the implication

$$A \in \bar{\mathscr{B}} \;\Rightarrow\; \forall B \in \bar{\mathscr{B}}_\mu,\; A \cap B \in \bar{\mathscr{B}}_\mu .$$

As for the last equality in (5), see [2, p. 14, Corollary].

Now let $\Lambda \in \mathscr{B}$. Then $\mathscr{B}$ is a σ-algebra, and so $\mathscr{A} =_{\mathrm{d}} \mathscr{B}^{\mathrm{loc}} = \mathscr{B}$. Hence (5) reduces to $\mathscr{B} = \bar{\mathscr{B}} = \mathscr{A}$. Also Λ satisfies the requirements imposed on Λ_0 in (e), and so $q = 1$. ∎

Lemma 6.5 does not provide a condition on the measure space $(\Lambda, \bar{\mathscr{B}}, \mu)$, which is necessary and sufficient to ensure that the indicator spectral measure has a preassigned multiplicity q. Indeed, it is doubtful if such a condition is available in so general a context. But for the measure space $(\Gamma, \mathrm{Bl}(\Gamma), m)$ occurring in (6.1), such a condition can be stated. In fact, q is completely determined by the topology of Γ, as we shall now show. This results from the fundamental fact, [10, p. 109] that the l.c.a. group Γ has the canonical decomposition:

$\Gamma = \bigcup_{\alpha \in \mathscr{I}} \Gamma_\alpha$, *where* $\Gamma_\alpha \in \mathscr{B}(\Gamma)$ *are* σ*-compact, clopen and disjoint, and* $\mathscr{I}$ *is an index set the cardinality of which is uniquely determined by the topology of* Γ. (6.6)[7]

From the property that $m(V) > 0$ for nonvoid open $V \subseteq \Gamma$, it follows that an open set V in $\mathscr{B}_m(\Gamma)$ can intersect at most a countable number of Γ_α. Hence by the outer regularity of m, the same holds for any $B \in \mathscr{B}_m(\Gamma)$; whence once again,

$$\forall B \in \mathscr{B}(\Gamma),\ B \textit{ intersects at most countably many } \Gamma_\alpha . \tag{6.7}$$

From (6.7) and the partial conditions on q obtained in Lemma 6.5 (d), (e), (f) we get the next theorem. Only part (a) of the theorem and the inequality $q > \aleph_0$ in part (b) are needed in this paper.

6.8 THEOREM. *With the notation in* (6.1), (6.2), *and* (6.6), *let* $\forall A \in \mathscr{A}(\Gamma)$, $E(A) = M_{\chi_A}$, *and* q *be the multiplicity of* $E(\cdot)$. *Then*

(a) $q = 1$, iff Γ *is* σ*-compact,*

(b) *either* $q = 1$, *or* $q =$ card $\mathscr{I} > \aleph_0$.

Proof. We appeal to Lemma 6.5, taking Λ, $\bar{\mathscr{B}}$, μ to be Γ, Bl(Γ), m, and exploit (6.7).

(a) Let Γ be σ-compact. Then of course $\Gamma \in \mathscr{B}(\Gamma)$. Hence by 6.5(f), $q = 1$.

Conversely, let $q = 1$. Then 6.5(e),

$$\exists \Gamma_0 \in \mathscr{B}(\Gamma) \ni \forall B \in \mathscr{B}(\Gamma), m(B \setminus \Gamma_0) = 0.$$

Since by (6.6) each $\Gamma_\alpha \in \mathscr{B}(\Gamma)$, therefore $m(\Gamma_\alpha \setminus \Gamma_0) = 0$. Since Γ_α is open, we see that $\forall \alpha \in \mathscr{I}$,

$$0 < m(\Gamma_\alpha) = m(\Gamma_\alpha \setminus \Gamma_0) + m(\Gamma_\alpha \cap \Gamma_0) = m(\Gamma_\alpha \cap \Gamma_0).$$

Thus, $\forall \alpha \in \mathscr{I}$, Γ_0 intersects Γ_α. But since $\Gamma_0 \in \mathscr{B}(\Gamma)$, therefore by (6.7), Γ_0 can intersect at most countably many Γ_α. It follows that $\mathscr{I}$ must be countable. This means (cf. (6.6)) that Γ is a countable union of σ-compact sets and is therefore itself σ-compact. Thus (a).

[7] Professor D. J. Lutzer has informed us that the decomposition (6.6) is known to hold for any paracompact, locally compact Hausdorff space Γ. From this fact it is clear that our Theorem 6.8 and Corollary 6.9 will survive for such spaces Γ and those measures μ on Bl(Γ) which resemble Haar measure in regard to regularity, positivity, and finiteness.

(b) Let $q \neq 1$. Then by (a), Γ is not σ-compact, and hence by (6.6),

$$r \underset{\mathrm{d}}{=} \operatorname{card} \mathscr{I} > \aleph_0 . \tag{1}$$

We now assert that $q \geqslant r$, i.e.,

$$G \subseteq \mathscr{L}_2 \,\&\, \operatorname{card} G < r \;\Rightarrow\; \mathscr{M}_G \underset{\mathrm{d}}{=} \mathfrak{S}\{E(A)(G) : A \in \mathscr{A}(\Gamma)\} \neq \mathscr{L}_2 . \tag{I}$$

Proof of (I). Let $G = \{g_j\}_{j\in J} \subseteq \mathscr{L}_2$, where

$$p \underset{\mathrm{d}}{=} \operatorname{card} J < r. \tag{2}$$

Each g_j, being in $\mathscr{L}_2$, has support $S_j \in \mathscr{B}(\Gamma)$. Hence by (6.7) each S_j intersects at most countably many Γ_α. Clearly therefore,

$$\bigcup_{j\in J} S_j \subseteq \bigcup_{\alpha\in\mathscr{I}_0} \Gamma_\alpha \underset{\mathrm{d}}{=} \Gamma_0 ,$$

where $\mathscr{I}_0 \subseteq \mathscr{I}$, and by (1) and (2),

$$\operatorname{card} \mathscr{I}_0 \leqslant p \cdot \aleph_0 < r \underset{\mathrm{d}}{=} \operatorname{card} \mathscr{I}. \tag{3}$$

Now for all A in $\mathscr{A}(\Gamma)$, $E(A)(g_j) = \chi_A \cdot g_j$ has its support inside S_j, and therefore inside Γ_0. The same must therefore be the case with all finite linear combinations $\sum_{i=1}^{n} a_i E(A_i)(g_{j_i})$ (where $a_i \in \mathbb{C}$, $A_i \in \mathscr{A}(\Gamma)$, $j_i \in J$) and with their limits. Thus

$$\forall f \in \mathscr{M}_G , \qquad S_f \subseteq \Gamma_0 . \tag{4}$$

Now by (3), $\mathscr{I} \setminus \mathscr{I}_0$ is nonvoid. Let $\alpha \in \mathscr{I} \setminus \mathscr{I}_0$, and f be any function in $\mathscr{L}_2$ with $S_f \subseteq \Gamma_\alpha$. Then since $\Gamma_\alpha \parallel \Gamma_0$, we have $S_f \parallel \Gamma_0$, whence by (4), $f \notin \mathscr{M}_G$. Thus (I) is proved.

For $q \neq 1$ we have shown that $\aleph_0 < r \leqslant q$. To complete the proof that $q = r$, we need only show that

$$\exists G = \{g_\alpha\}_{\alpha\in\mathscr{I}} \subseteq \mathscr{L}_2 \ni \mathscr{M}_G \underset{\mathrm{d}}{=} \mathfrak{S}\{E(A)(G) : A \in \mathscr{A}(\Gamma)\} = \mathscr{L}_2 . \tag{II}$$

Proof of (II). Let $\forall \alpha \in \mathscr{I}$, $\mathscr{L}_2^{(\alpha)} =_{\mathrm{d}} \{f : f \in \mathscr{L}_2 \,\&\, S_f \subseteq \Gamma_\alpha\}$. Then from (6.6) it follows that

$$\mathscr{L}_2 = \sum_{\alpha\in\mathscr{I}} \mathscr{L}_2^{(\alpha)}, \qquad \mathscr{L}_2^{(\alpha)} \perp \mathscr{L}_2^{(\beta)}, \qquad \alpha \neq \beta. \tag{5}$$

Obviously, $\mathscr{L}_2^{(\alpha)}$ is isometrically isomorphic to

$$L_2^{(\alpha)} \underset{\mathrm{d}}{=} L_2(\Gamma_\alpha , \mathrm{Bl}(\Gamma_\alpha), m_\alpha ; \mathbb{C}),$$

where m_α is the restriction of m to $\mathrm{Bl}(\Gamma_\alpha)$. Fix α, and apply Lemma 6.5 to the space $L_2^{(\alpha)}$, i.e., in 6.5 take

$$\Lambda \underset{\mathrm{d}}{=} \Gamma_\alpha, \qquad \bar{\mathscr{B}} \underset{\mathrm{d}}{=} \mathrm{Bl}(\Gamma_\alpha), \qquad \mu \underset{\mathrm{d}}{=} m_\alpha .$$

Then, as is easily verified (cf. (6.2), 6.5(ii)),

$$\bar{\mathscr{B}}_\mu = \mathscr{B}_m(\Gamma) \cap 2^{\Gamma_\alpha}, \qquad \mathscr{B} = \mathscr{B}(\Gamma) \cap 2^{\Gamma_\alpha}, \qquad \mathscr{A} = \mathscr{A}(\Gamma) \cap 2^{\Gamma_\alpha}. \quad (6)$$

Now define

$$\forall A \in \mathscr{A}, \qquad E_\alpha(A) = M_{\chi_A} \underset{\mathrm{d}}{=} E(A). \quad (7)$$

Then the results 6.5(a)–(f) are valid for $E_\alpha(\cdot)$. Thus

$$E_\alpha(\cdot) \text{ is a spectral measure for } L_2^{(\alpha)} \text{ on } \mathscr{A}. \quad (8)$$

Also, since by (6.6), $\Gamma_\alpha \in \mathscr{B}(\Gamma)$, therefore by (6), $\Gamma_\alpha \in \mathscr{B}$. Hence by 6.5(f), E_α has multiplicity 1. Hence

$$\exists f_\alpha \in L_2^{(\alpha)} \ni \mathfrak{S}\{E(A)(f_\alpha) : A \in \mathscr{A}\} = L_2^{(\alpha)}. \quad (9)$$

Now let

$$g_\alpha \underset{\mathrm{d}}{=} \begin{cases} f_\alpha , & \text{on } \Gamma_\alpha , \\ 0, & \text{on } \Gamma \setminus \Gamma_\alpha . \end{cases} \quad (10)$$

Then clearly from (7), (9), and (6),

$$\mathscr{L}_2^{(\alpha)} = \mathfrak{S}\{E(A)(g_\alpha) : A \in \mathscr{A}\} = \mathfrak{S}\{E(A)(g_\alpha) : A \in \mathscr{A}(\Gamma) \cap 2^{\Gamma_\alpha}\}. \quad (11)$$

Indeed, we claim that

$$\mathscr{L}_2^{(\alpha)} = \mathfrak{S}\{E(A)(g_\alpha) : A \in \mathscr{A}(\Gamma)\}. \quad (12)$$

For consider any $A \in \mathscr{A}(\Gamma)$. By (7) and (10),

$$E(A \setminus \Gamma_\alpha)(g_\alpha) = \chi_{A \setminus \Gamma_\alpha} \cdot g_\alpha = 0,$$

and so

$$E(A)(g_\alpha) = \{E(A \cap \Gamma_\alpha) + E(A \setminus \Gamma_\alpha)\}(g_\alpha) = E(A \cap \Gamma_\alpha)(g_\alpha).$$

Since $A \cap \Gamma_\alpha \in \mathscr{A}(\Gamma) \cap \mathscr{L}^{\Gamma_\alpha}$, (12) follows from (11).

Equation (12) holds, of course, for all $\alpha \in \mathscr{I}$. Hence by (5),

$$\begin{aligned} \mathscr{L}_2 &= \sum_{\alpha \in \mathscr{I}} \mathscr{L}_2^{(\alpha)} = \sum_{\alpha \in \mathscr{I}} \mathfrak{S}\{E(A)(g_\alpha) : A \in \mathscr{A}(\Gamma)\} \\ &= \mathfrak{S}\{E(A)(G) : A \in \mathscr{A}(\Gamma)\}, \qquad \text{where} \quad G \underset{\mathrm{d}}{=} \{g_\alpha\}_{\alpha \in \mathscr{I}} . \end{aligned}$$

This establishes (II) and completes the proof of (b). ∎

Theorem 6.8 does not quite fit our needs, since the unitary representations of Γ have spectral measures on the σ-algebra $\mathrm{Bl}(\hat{\Gamma})$, which will be smaller than $\mathscr{A}(\hat{\Gamma})$ for non-σ-compact Γ (cf. 5.3(a), (6.3)). We have therefore to obtain a corresponding result for the indicator spectral measure restricted to $\mathrm{Bl}(\Gamma)$ in order to use the results of Section 5 in the present setting. This result is an easy corollary:

6.9 COROLLARY. *With the notation in* (6.1) *and* (6.6), *let* $\forall B \in \mathrm{Bl}(\Gamma)$, $E(B) = M_{\chi_B}$, *and q be the multiplicity of* $E(\cdot)$. *Then*

(a) $E(\cdot)$ *is a spectral measure for* $\mathscr{L}_2$ *on* $\mathrm{Bl}(\Gamma)$,

(b) $q = 1$, iff Γ *is σ-compact*,

(c) *either* $q = 1$ *or* $q > \aleph_0$.

Proof. (a) is trivial. (b) With the notation in (6.2), let $\forall A \in \mathscr{A}(\Gamma)$, $E_1(A) = M_{\chi_A}$, and let q_1 be the multiplicity of $E_1(\cdot)$. Since $E(\cdot)$ is a restriction of $E_1(\cdot)$, it follows that for all $G \subseteq \mathscr{L}_2$,

$$\{E(B)(G) : B \in \mathrm{Bl}(\Gamma)\} \subseteq \{E_1(A)(G) : A \in \mathscr{A}(\Gamma)\},$$

whence (cf. Definition 1.1)

$$1 \leqslant q_1 \leqslant q. \tag{1}$$

Now if $q = 1$, then by (1), $q_1 = 1$, and so by Theorem 6.8(a), Γ is σ-compect. Conversely, if Γ is σ-compact, then by Theorem 6.8(a), $q_1 = 1$. Also $\Gamma \in \mathscr{B}(\Gamma)$, and hence by (6.2) and (6.3), $\mathscr{B}(\Gamma) = \mathrm{Bl}(\Gamma) = \mathscr{A}(\Gamma)$. Thus $E(\cdot) = E_1(\cdot)$, and so $q = q_1 = 1$. Thus (b).

(c) Let $q \neq 1$. Then by (b), Γ is not σ-compact. Hence by (1) and Theorem 6.8(b), $q \geqslant q_1 > \aleph_0$. ∎

We now turn to the group $(M_\lambda : \lambda \in \hat{\Gamma})$. It is well known that this is a strongly continuous group of unitary operators on $\mathscr{L}_2$ onto $\mathscr{L}_2$, and it is trivial to check that

the spectral measure $E(\cdot)$ *of* $(M_\lambda : \lambda \in \hat{\Gamma})$ *is given by* $E(B) = M_{\chi_B}$, $B \in \mathrm{Bl}(\Gamma)$. (6.10)

By 6.9(b), this $E(\cdot)$ has multiplicity 1, iff Γ is σ-compact. Hence by 5.5,

for σ-compact Γ, the unitary group $(M_\lambda : \lambda \in \hat{\Gamma})$ *has multiplicity* 1. (6.11)

In the σ-compact case our Corollary 5.10 is therefore applicable and yields the following result.

6.12 THEOREM. *With the notation* 5.1(i)–(iv), *and* (6.1), *let* (i) Γ *be σ-compact,* (ii) T *be a closed linear operator on* $\mathscr{L}_2$ *to* $\mathscr{L}_2$ *with domain* $\mathscr{D}_T$ e.d. *in* $\mathscr{L}_2$, (iii) $\forall \lambda \in \hat{\Gamma}$, $M_\lambda \cdot T = T \cdot M_\lambda$. *Then there exists a Borel measurable function* ϕ *on* Γ *to* $\mathbb{C}$ *such that* $T = M_\phi$.

Proof. By (i) and (6.11) the group $(M_\lambda : \lambda \in \hat{\Gamma})$ has multiplicity 1. Hence by (ii), (iii), and Corollary 5.10, T is an E-integral, say $\int_\Lambda \phi(t)\, E(dt)$, where ϕ is Borel measurable on Λ to $\mathbb{C}$, and by (6.10), $E(\cdot)$ is the indicator spectral measure. By 6.5(b) the last integral is M_ϕ; hence $T = M_\phi$. ▮

From Theorem 6.12 we can deduce a generalized version of Bochner's Theorem [1] by Fourier–Plancherel (FP) transformation. Let

$$\hat{\mathscr{L}}_2 \underset{\mathrm{d}}{=} L_2(\hat{\Gamma}, \mathrm{Bl}(\hat{\Gamma}), \hat{m}; \mathbb{C}),$$

where $\hat{m}$ is the Haar measure for $\hat{\Gamma}$ dual to m (cf. [14, 3.17]), and let V be the FP transformation on $\mathscr{L}_2$ onto $\hat{\mathscr{L}}_2$. Then (cf., e.g., [14, 5.6]),

$$\forall t \in \Gamma, \qquad \tau_t = V^* M_{[t,\cdot]} V. \tag{6.13}$$

We now assert the following:

6.14 GENERALIZED BOCHNER THEOREM. With the notation 5.1(i)–(iv), (6.1), *and* (6.2), *let* (i) $\hat{\Gamma}$ *be σ-compact,* (ii) S *be a closed linear operator from* $\mathscr{L}_2$ *to* $\mathscr{L}_2$ *with domain* $\mathscr{D}_S$ e.d. *in* $\mathscr{L}_2$, (iii) $\forall t \in \Gamma$, $\tau_t \cdot S = S \cdot \tau_t$. *Then there exists a* $\mathrm{Bl}(\hat{\Gamma})$ *measurable function* ϕ *on* $\hat{\Gamma}$ *to* $\mathbb{C}$ *such that*

$$\forall f \in \mathscr{D}_S, \qquad (Sf)^\wedge(\lambda) = \phi(\lambda) \cdot \hat{f}(\lambda), \qquad \text{a.e. } \hat{m} \text{ on } \hat{\Gamma},$$

where $\hat{f}$ *is the* FP *transform* $V(f)$ *of* f, *and* $\hat{m}$ *is any Haar measure for* $\hat{\Gamma}$.

Proof. Since the FP transformation V is unitary on $\mathscr{L}_2$ onto $\hat{\mathscr{L}}_2$, it follows from (ii) that

$$T \underset{\mathrm{d}}{=} VSV^* \text{ is a closed linear operator on } \hat{\mathscr{L}}_2 \text{ to } \hat{\mathscr{L}}_2 \text{ with domain } \mathscr{D}_T \text{ e.d. in } \hat{\mathscr{L}}_2. \tag{1}$$

Next from the definition of T, (6.13), and (iii),

$$\forall t \in \Gamma, \qquad M_{[t,\cdot]} \cdot T = T \cdot M_{[t,\cdot]}. \tag{2}$$

It follows from (i), (1), (2), and Theorem 6.12 that $T = M_\phi$, where ϕ is $\mathrm{Bl}(\hat{\Gamma})$-measurable on $\hat{\Gamma}$ to $\mathbb{C}$. This means that for all f in $\mathscr{L}_2$,

$$(Sf)^\wedge = VS(f) = (VSV^*)(Vf) = T(\hat{f})) = \phi \cdot \hat{f}, \qquad \text{a.e. } \hat{m}. \quad ▮$$

Theorem 6.14 specializes to Bochner's classical theorem [1] on taking $\Gamma = \mathbb{R}$ and S continuous on $L_2(\mathbb{R})$, the resulting ϕ being then in $L_\infty(\mathbb{R})$. Theorems 6.12 and 6.14 have generalizations also: the hypotheses that Γ, $\hat{\Gamma}$ are σ-compact can be removed, if we are willing to allow the functions ϕ to be $\mathscr{A}(\Gamma)$- or $\mathscr{A}(\hat{\Gamma})$-measurable rather than $\mathrm{Bl}(\Gamma)$- or $\mathrm{Bl}(\hat{\Gamma})$-measurable. This generalization of 6.12 reads as follows.

6.15 THEOREM. *With the notation* 5.1(i)–(iv), (6.1), *and* (6.2), *let* (i) T *be a closed linear operator from* $\mathscr{L}_2$ *to* $\mathscr{L}_2$ *with domain* $\mathscr{D}_T$ e.d. *in* $\mathscr{L}_2$, (ii) $\forall\lambda \in \hat{\Gamma}$, $M_\lambda \cdot T = T \cdot M_\lambda$. *Then there exists an* $\mathscr{A}(\Gamma)$-*measurable function* ϕ *on* Γ *to* $\mathbb{C}$ *such that* $T = M_\phi$.

Theorem 6.14 admits an analogous generalization. We shall not present the proofs here, as these theorems fall outside the scope of this paper cf. Sect. 1.[8] They depend on the special circumstance that the Hilbert space $\mathscr{L}_2$ has two naturally attached σ algebras, viz $\mathrm{Bl}(\Gamma)$ and $\mathscr{A}(\Gamma)$, whereas for general $\mathscr{H}$ just one σ-algebra $\mathscr{B}$ is given. We shall only mention that to prove 6.15 we apply 6.12 to each of the σ-compact components $L_2^{(\alpha)} = L_2(\Gamma_\alpha, \mathrm{Bl}(\Gamma_\alpha), m_\alpha; \mathbb{C})$ appearing in the proof of Theorem 6.8(b), and piece together the resulting functions ϕ_α (supported on Γ_α) to form a single function ϕ, which is $\mathscr{A}(\Gamma)$-measurable.

Another extension of Theorem 6.14, due to Foures and Segal [3, Theorem 1], is obtained by replacing the "scalar" $\mathscr{L}_2$ space of (6.1) by the "vectorial" $\mathscr{L}_2$ space

$$\mathscr{L}_2 = L_2(\Gamma, \mathrm{Bl}(\Gamma), m; W),$$

where W is a separable Hilbert space, and for $\hat{m}$ almost all λ in $\hat{\Gamma}$, $\phi(\lambda)$ is a closed, densely defined linear operator from W to W. In [16] we have shown that it is not just the Bochner Theorem 6.14 (a specific case of our Corollary 5.10) which admits such an operatorial extension: the more general Theorem 1.5 and Corollaries 4.1, 4.4 do likewise.

7. THE CLASSICAL COMMUTANT CONDITION

Specializing further, we shall now suppose that

$$\begin{aligned}&E(\cdot) \textit{ is a spectral measure for } \mathscr{H} \textit{ on } \mathrm{Bl}(\mathbb{R}) \textit{ of total multiplicity}\\ &q \leqslant \aleph_0, \textit{ and } H \underset{\mathrm{d}}{=} \textstyle\int_{\mathbb{R}} uE(du).\end{aligned} \tag{7.1}$$

[8] The proofs are given in [18].

For any operator S from $\mathscr{H}$ to $\mathscr{H}$, let

$$\{S\}' \underset{\mathrm{d}}{=} \{K : K \textit{ is a continuous linear operator on } \mathscr{H} \textit{ to } \mathscr{H} \textit{ and } K \cdot S \subseteq S \cdot K\}. \tag{7.2}$$

We shall now deduce the following classical theorem referred to in [20, p. 351] and [26, p. 191]:

7.3 THEOREM. *Let* (i) E *and* H *be as in* (7.1), *and* (ii) T *be a closed linear operator from* $\mathscr{H}$ *to* $\mathscr{H}$ *with domain* $\mathscr{D}_T$ c.d. *in* $\mathscr{H}$. *Then* T *is an* E*-integral*, iff $\{H\}' \subseteq \{T\}'$.

Proof. Let $\{H\}' \subseteq \{T\}'$. By Lemma 2.9(b), $\forall x \in \mathscr{H}$, $L_x \in \{H\}'$. Hence $\forall x \in \mathscr{H}$, $L_x \in \{T\}'$, i.e., $L_x \cdot T \subseteq T \cdot L_x$. Certainly therefore, T is E-reducing (cf. 1.4(d)). Hence, by the $(\gamma) \Rightarrow (\alpha)$ part of Theorem 1.5, T is an E-integral.

Conversely, let T be an E-integral, say $T = \int_{\mathbb{R}} \phi(u)\, E(du)$. If $K \in \{H\}'$, then by (7.2), K is continuous on $\mathscr{H}$ and $K \cdot H \subseteq H \cdot K$. Hence by Triviality 5.6, $K \cdot T \subseteq T \cdot K$; i.e., $K \in \{T\}'$. Thus $\{H\}' \subseteq \{T\}'$. ∎

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